Minimum traveltime calculations in anisotropic media using graph theory

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Summary

The efficient modeling of anisotropic wave propogation is an issue of growing importance due to the need for accurate traveltime tables in subsurface imaging algorithms. We propose an adaptation of the shortest path ray tracing (SPR) method that effectively calculates first arrival times in transversely isotropic media, the most commonly modeled form of anisotropy. The modification does not change the asymptotic complexity of the SPR algorithm and allows more accurate travel-time aproximations with only a minimal increase in compute time. The method calculates edge weights for the search graph by using the edge's orientation and an anisotropy function relating group angle to group velocity. Once anisotropic weights are assigned to all of the edges in the graph, Djikstra's shortest path algorithm can be applied to determine the first-arrival times for every node in the graph. The method was initially tested on several models containing elliptically anisotropic layers: we then generalized the code to account for transversely isotropic media using Thomsen's expressions for weak anisotropy. For homogenous anisotropic media, comparisons between SPR generated traveltimes and analytically determined traveltimes show excellent agreement. Traveltimes for several larger inhomgenous models are also included.

Introduction

Shortest path ray tracing (SPR) is a simple application of Fermat's principle; SPR relies on the assumption that rays follow paths such as to minimize traveltime. The first step of modeling P-wave propogation using SPR is descritizing the subsurface into a regular network of nodes and connecting each node to local neighbors. Traveltime are then assigned to each edge; these values represent the time required to travel from one node to a neighbor. After the graph representing the velocity model is initialized, a shortest-path algorithm is applied to determine the minimum traveltime between an arbitrary source node and every other node within the network. The most common methods used to determine shortest paths are variants of Djikstra's algorithm using advanced data structures.

Recent experiments applying graph theory towards the modelling of seismic wave propogation (Moser, 1991) (Fischer and Lees, 1993) (Cheng and House, 1996) have shown SPR to be a robust method for obtaining first-arrival traveltimes for isotropic media in 2 and 3 dimensions. Unlike ray bending methods, SPR is guaranteed to find a global minimum traveltime instead of possibly converging on local features. SPR also avoids the traditional problems of ray shooting techniques and can provide accurate times for diffracted rays and times within shadow zones. SPR's weakest point is the high computational cost of the shortest-path calculation: a naive implementation of Djikstra's algorithm can make modeling too slow to process reasonably sized graphs. However, advances in shortest path algorithms (Goldberg et al., 1993) seem to promise much higher speeds if creatively adapted for SPR. Although this paper examines work in 2D, SPR algorithms trivially extend to 3D due to the nature of graph representations.

Another interesting property of graph theoretic modelling is the ease with which it can be adapted to handle anisotropic models; situations where ray velocity is dependant upon orientation. In previous SPR algorithms, node-to-node traveltimes have been calculated by dividing the cartesian length of a given edge by the velocity of the medium through which it passes. Two additional steps in the graph construction phase allow SPR to be generalized to transversely isotropic media. First, computation of the direction of every edge is necessary. These directions are then converted into velocities using knowledge of the polar functions relating group angle to group velocity for every zone of differing anisotropic parameters within the model.

Graph Construction And Complexity

SPR algorithms can generally be broken into two stages, a pre-processing stage that creates the graph and generates edge weights from the velocity model, and the actual tracing phase that determines the shortest paths from a particular source.

The tracing phase is usually implemented using Djikstra's algorithm, with the choice of data structure for the priority que determining asymptotic complexity. Implementations using heaps are of $O(E \log N)$ if designed to correctly exploit partial ordering, O(x) meaning "on the order of x" in the notation of complexity theory. E and N represent the number of edges and nodes in the graph respectively. Naive implementations that depend on resorting linear arrays are generally $O(N^2)$. SPR studies in the literature have limited themselves to these algorithms. However, implementations significantly faster do exist; one example exploits fibonnaci heaps to achieve $O(N \log N + E) = O(N \log N)$.

These complexity expressions become more meaningful if we examine the size and sparsity of the graphs used for tracing. We descritize the velocity model into an n x m grid with some number of nodes being placed on the edge of each cell (NPE). Each node is connected to all nodes that are within adjoining cells and not colinear along cell boundaries. This design has been well-explored by Fischer and Lee (1993): they estimate that between 2 - 4 NPE are required for accurate tracing, depending upon the optimizations and error correction methods used. We can easily derive expression relating the dimension of the grid, the NPE count and the total number of nodes (N_t) and edges (E_t).

$$N_t = NPE(2nm + n + m) \tag{1}$$

$$E_t = 12(NPE)^2(nm) \tag{2}$$

The key step in adapting a velocity model for use by SPR methods is the mapping of velocities from the first model into traveltime weights on the SPR graph. Previous studies (Moser, 1991) (Fischer and Lees, 1993) have used isotropic models: the weight for a graph edge is calculated by dividing the distance between the nodes by the velocity of the cell containing the link.

For an anisotropic model, the direction of the edge and the anisotropy parameters for the cell are also required for computation of the edge weight. We attached coordinate information to each node in the graph to allow the direction of edges to be easily determined. Evaluation of the function relating group angle to group velocity proved to be the most expnesive part of the weight-determination step.

The construction phase for anisotropic models will, in the worst case, be proportional to E_t: in an irregular graph, the velocity function might have to be evaluated for every edge. However, since the graphs we examine are strictly regular, a caching system which stores the results of previous evaluations can reduce this to $O(M)$ where M is the number of distinct media in the graph. $O(E_t)$ will be small in comparison to the ray-tracing time for any significant value of N_t while M is usually almost insignificant. Our anisotropic SPR code spent less then 0.1% of its runtime constructing the graph and determining weights for graphs with aproximately 3000 nodes. Two preprocessing modules were developed, one to handle elliptical anisotropy and a second for modeling weak TI media.

Elliptical Anisotropy

Although elliptical anisotropy is a rare phenomenon in p-wave propogation (Helbig, 1983), the simplicity of the expressions describing it make this type of anisotropy an attractive test for modeling algorithms. Elliptical anisotropy for p-waves occurs only when

$$(C_{11} - C_{44})(C_{33} - C_{44}) - (C_{13} + C_{44})^2 = 0. \tag{3}$$

$C_{\alpha\beta}$ are components of the 6 x 6 elastic modulus matrix. This relation will not be satisfied by media possessing anisotropy caused by fine layering (Berryman, 1979).

We follow Levin's derivation of the phase velocity of p-waves(v)with respect to phase angle (θ) (Levin, 1978). The relevant parameters for elliptical anisotropy are the horizontal and vertical velocity of p-waves in the medium, v_h and v_v respectively. V denotes group velocity while v denotes represents phase velocity.

$$v(\theta) = \sqrt{v_v^2 cos^2\theta + v_h^2 sin^2\theta} \tag{4}$$

The relation between θ, v, group velocity (V), and group angle (ϕ) can be determined via geometric construction.

$$V(\phi) = \frac{v(\theta)}{cos(\phi - \theta)} \tag{5}$$

Levin derives an expression for θ with respect to ϕ.

$$\theta = tan^{-1}\left[\frac{tan\phi}{(\frac{v_h}{v_v})^2}\right] \tag{6}$$

Substitution into equation 4 yields an expression for group velocity with respect to ϕ.

$$V(\phi) = \sqrt{\frac{(v_v v_h)^2}{v_v^2 sin^2\phi + v_h^2 cos^2\phi}} \tag{7}$$

Weak TI anisotropy

Many solids possess transversely isotropic properties that cannot be accurately aproximated using the simple equations for elliptical anisptropy. The full equations for group velocity in a TI medium are fairly complex: we decided to use the weak forms which hold for values of δ much less then one, δ being the most important of the four anisotropic parameters provided by Thomsen (Thomsen, 1986). The other relevant paramters are ε, the second measure of P-wave anisotropy in TI media and α_o, the vertical p-wave velocity. We follow his derivation of the group velocity expressions. The equation relating phase angle to phase velocity is

$$v = \alpha_o(1 + \delta sin^2\theta cos^2\theta + \varepsilon sin^4\theta) \tag{8}$$

Although the calculation of phase velocity from phase angle is fairly simple, the determination of group velocities is not quite as easy. The group velocity for p-waves can be expressed as ...

$$V(\phi) = v(\theta)\Big[1 + \frac{1}{2v^2}[\frac{\partial v}{\partial \theta}]^2\Big] \quad (9)$$

We used a linearized aproximation of the group velocity expression.

$$V(\phi) = v(\theta) \quad (10)$$

This expression states that the group velocity for a given ϕ (group angle) is equivalent to the phase velocity for an associated θ (phase angle) specified by the following relation...

$$\phi = tan^{-1}\Big[tan\theta[1 + 2\delta + 4(\varepsilon - \delta)sin^2\theta]\Big] \quad (11)$$

We used a modified bisection algorithm to solve the above relation for phase angle given group angle. Because the bisection algorithm is somewhat slow, a caching system was used to store $\phi_p \Rightarrow \theta_p$ relations already computed, minimizing the number of times equation 11 must be solved. Bisection is guaranteed to converge if proper initial bounds for the result can be given. Other approaches that were considered included a truncated Fourier-type cosine aproximation of the group velocity (Faria and Stoffa, 1994) and a tedious closed-form relating θ in terms of ϕ, based upon the general closed form for quartic equations.

Examples

The first tests were performed on very simple elliptically anisotropic media (Fig.1). The resulting traveltime images clearly show anisotropic propogation. A simple program was also written to analytically compute the traveltimes for a homogenous anisotropic medium given the group velocity function. This was accomplished by taking the cartesian distance between the source and every grid point and then dividing the group velocity for the angle of each segment by the length of the segment. All of the models shown below were computed on a coarse 30 x 30 grid with 4 nodes per graph edge. The traveltime contours are depicted in intervals of 1 second.

We later tested the SPR algorithm on several homogenous media with anisotropy parameters equivalent to real rocks discussed by Thomsen (Thomsen, 1986). Pierre Shale at 5000psi was the most anisotropic medium modeled (Fig.2).

Faria and Stoffa tested their TI traveltime compuation scheme on the same model and produced similar results (Faria and Stoffa, 1994). We also tested two heterogenous models. The first model (Fig.3) was composed of 4 layers possessing transversely isotropic properties. The thickness of each layer is notated as S_t.

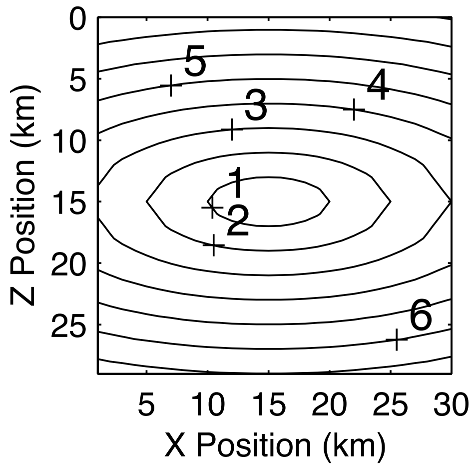

FIG. 1. Modeled traveltimes in a homogenous elliptically anisotropic medium: $V_v = 2km/s$ and $V_h = 5km/s$

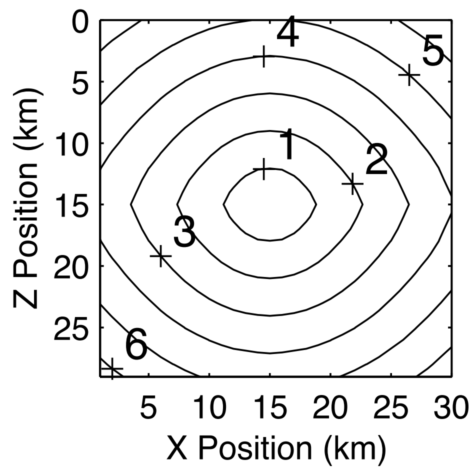

FIG. 2. Modeled traveltimes for a homogenous TI medium with attributes identical to the Pierre Shale at 5000 psi (Thomsen, 1986) .

Heterogenous TI Model - Layer Parameters

1. Cotton Valley Shale, $\alpha = 2.890$, $\delta = 0.205$, $\varepsilon = 0.135$, $S_t = 8$
2. Pierre Shale at 5000psi, $\alpha = 3.048$, $\delta = -0.050$, $\varepsilon = .255$, $S_t = 6$
3. Taylor Sandstone, $\alpha = 3.368$, $\delta = -0.035$, $\varepsilon = .110$, $S_t = 9$
4. Ft. Taylor Siltstone, $\alpha = 4.877$, $\delta = -0.045$, $\varepsilon = .045$, $S_t = 7$

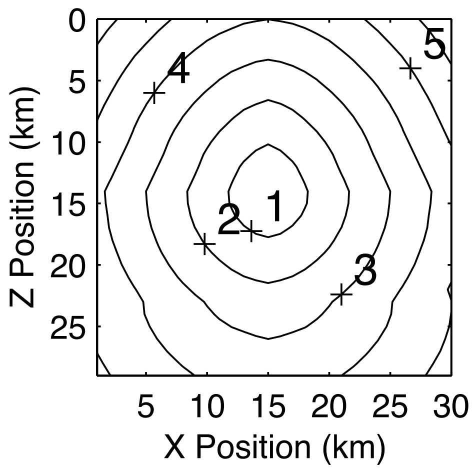

FIG. 3. SPR traveltimes for a 4-layer TI model

We also claculated the traveltimes for a 4-layer elliptically anisotropic model.

Discussion and Conclusions

The analytic traveltimes (not shown) and the modeled traveltimes calculated for the homogenous models agree closely. One should note that since both the analytic and SPR packages use the same group velocity compuation subroutines the comparison only detects errors due to angular and spatial descritization in the modeling process. The average error across the Pierre Shale times was aproximately 0.5%. Comparisons of spatial error plots for several media at different NPE counts reveal that both the distribution of nodes and the type of anisotropy effect accuracy. Extreme anisotropy produced the greatest (still under 0.8 %) errors. Xu and Lathrop (1994) produced an examination of this problem in the context anisotropic forest-fire spread and GIS systems.

Shortest path ray tracing requires only simple modifications for use on anisotropic models. The only significant addition needed is a module that computes group velocity from group angle for the particular type of anisotropic medium. Because these modifications are part of the graph construction phase and not the path-finding phase, the additional time required to model anisotropic media is, in worse case, linearly related to the number of edges in the graph. Since the algorithms currently being used for determining shortest paths run in either $O(V^2)$ or $O(E/logV)$, the time to initialize an anisotropic graph will be small in comparison to the tracing time. Traveltime determinations for the method are accurate for the case of homogenous and simple heterogenous materials: evaluation of the accuracy of the method on larger, more complex models is an area for future investigation.

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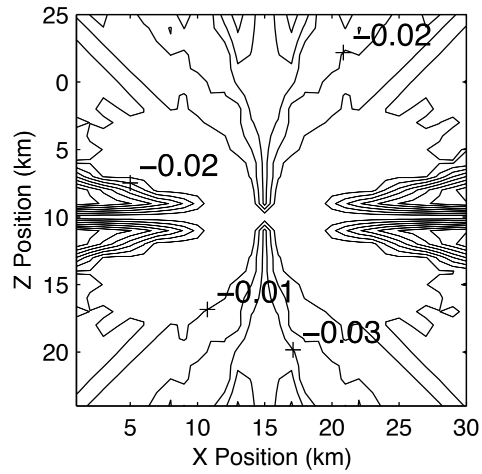

FIG. 4. A spatial plot of absolute traveltime error for the Pierre Shale model. These values were calculated by computing the time difference between the analytic and SPR results. The contour interval is .01 seconds.

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